

On the time-dependent grade-two model for the magnetohydrodynamic flow: 2D case

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Abstract

In this paper we discuss the MHD flow of a second grade fluid, in particular we prove the existence and uniqueness of a weak solution of a time-dependent grade two fluid model in a two-dimensional Lipschitz domain. We follow the methodology of [3], i.e., we use a constructive method which can be adapted to the numerical analysis of finite-element schemes for solving this problem numerically.

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1 Introduction

A fluid of grade two is a non-Newtonian fluid of differential type introduced by Rivlin and Ericksen in [8]. An analysis in [1] shows that the equation of a fluid of grade two is given by

$$\frac{\partial}{\partial t}(\mathbf{u} - \alpha \Delta \mathbf{u}) - \nu \Delta \mathbf{u} + \sum_j (\mathbf{u} - \alpha \Delta \mathbf{u})_j \nabla u_j - \mathbf{u} \cdot \nabla (\mathbf{u} - \alpha \Delta \mathbf{u}) = -\nabla p + \mathbf{f}$$

$$\operatorname{div} \mathbf{u} = 0$$

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where $\alpha \geq 0$ is a constant of material, $\nu > 0$ is the viscosity of the fluid, \mathbf{u} is the velocity field, and p is pressure. For $\alpha = 0$ the classical Navier-Stokes equation is obtained.

On the other hand, in several situations the motion of incompressible electrical conducting fluid can be modeled by the magnetohydrodynamic equation, which correspond to the Navier-Stokes equations coupled with the Maxwell equations. In presence of a free motion of heavy ions, not directly due to the electrical field (see Schluter [4] and Pikelnier [3]), the MHD equation can be reduced to

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \frac{\nu}{\rho_m} \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \frac{\mu}{\rho_m} \mathbf{h} \cdot \nabla \mathbf{h} &= \mathbf{f} - \frac{1}{\rho_m} \nabla (p^* + \frac{\mu}{2} \mathbf{h}^2) \\ \frac{\partial \mathbf{h}}{\partial t} - \frac{1}{\mu \sigma} \Delta \mathbf{h} + \mathbf{u} \cdot \nabla \mathbf{h} - \mathbf{h} \cdot \nabla \mathbf{u} &= -\text{grad } \omega \end{aligned} \quad (1)$$

$$\text{div } \mathbf{u} = \text{div } \mathbf{h} = 0$$

with

$$\mathbf{u}|_{\partial\Omega} = \mathbf{h}|_{\partial\Omega} = 0. \quad (2)$$

Here, \mathbf{u} and \mathbf{h} are respectively the unknown velocity and magnetic field; p^* is the unknown hydrostatic pressure; ω is an unknown function related to the heavy ions (in such way that the density of electric current, j_0 , generated by this motion satisfies the relation $\text{rot } j_0 = -\sigma \nabla \omega$) is the density of mass of the fluid (assumed to be a positive constant); $\mu > 0$ is the constant magnetic permeability of the medium; $\sigma > 0$ is the constant electric conductivity; $\nu > 0$ is the constant viscosity of the fluid; \mathbf{f} is a given external force field.

In the case the MHD equation coupled with the equation of an incompressible second grade fluid, the model can be write as

$$\begin{aligned} \frac{\partial}{\partial t} (\mathbf{u} - \alpha \Delta \mathbf{u}) - \nu \Delta \mathbf{u} + \text{curl} (\mathbf{u} - \alpha \Delta \mathbf{u}) \times \mathbf{u} - (\mathbf{h} \cdot \nabla) \mathbf{h} &= \mathbf{f} - \nabla (p^* + \mathbf{h}^2) \\ \frac{\partial \mathbf{h}}{\partial t} - \Delta \mathbf{h} + (\mathbf{u} \cdot \nabla) \mathbf{h} - (\mathbf{h} \cdot \nabla) \mathbf{u} &= -\text{grad } \omega \end{aligned} \quad (3)$$

$$\text{div } \mathbf{u} = \text{div } \mathbf{h} = 0$$

with

$$\mathbf{u}|_{\partial\Omega} = \mathbf{h}|_{\partial\Omega} = 0. \quad (4)$$

Note that when $\alpha = 0$ we recover the model (1).

One of the first mathematical results for this model type appears in [2], they prove the existence and uniqueness of solutions for a small time and global existence of solutions for small initial data in a conducting domain of \mathbb{R}^3 , based on the iterative scheme where discretization is performed in the spatial variables. In this paper we discuss the MHD flow of a second grade fluid, in particular we prove the existence and uniqueness of a weak solution of a time-dependent

grade two fluid model in a two-dimensional Lipschitz domain, where we follow the methodology of [3], i.e , we use semi-discretization in time and the work is in a domain of \mathbb{R}^2 .

2 Preliminary results

2.1 Notation

Let (k_1, k_2) denote a pair of non-negative integers, set $|k| = k_1 + k_2$ and define the partial derivative ∂^k by

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} \partial x_2^{k_2}}.$$

Then, for any non-negative integer m and number $r \geq 1$, recall the classical Sobolev space

$$W^{m,r}(\Omega) = \left\{ v \in L^r(\Omega); \partial^k v \in L^r(\Omega) \forall |k| \leq m \right\},$$

equipped with the seminorm

$$|v|_{W^{m,r}(\Omega)} = \left[\sum_{|k|=m} \int_{\Omega} |\partial^k v|^r dx \right]^{1/r},$$

and norm (for which it is a Banach space)

$$\|v\|_{W^{m,r}(\Omega)} = \left[\sum_{0 \leq |k| \leq m} \int_{\Omega} |v|_{W^{k,r}(\Omega)}^r \right]^{1/r},$$

with the usual extension when $r = \infty$. When $r = 2$, this space is the Hilbert space $H^m(\Omega)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation, but with the following modification for the norms in the non-Hilbert case. Let $\mathbf{u} = (u_1, u_2)$; then we set

$$\|\mathbf{u}\|_{L^r(\Omega)} = \left[\int_{\Omega} \|\mathbf{u}(x)\|^r \right]^{1/r},$$

where $\|\cdot\|$ denotes the Euclidean vector norm.

For functions that vanish on the boundary, we define for any $r \geq 1$,

$$W_0^{1,r}(\Omega) = \{ v \in W^{1,r}(\Omega); v|_{\partial\Omega} = 0 \}$$

and recall Poincaré's inequality, there exists a constant \mathcal{P} such that

$$\forall v \in H_0^1(\Omega), \quad \|v\|_{L^r(\Omega)} \leq \mathcal{P} |v|_{H^1(\Omega)}. \quad (5)$$

More generally, recall the inequalities of Sobolev embeddings in two dimension, for each $r \in [2, \infty)$, there exists a constant S_r such that

$$\forall v \in H_0^1(\Omega), \quad \|v\|_{L^r(\Omega)} \leq \mathcal{S}_r |v|_{H^1(\Omega)}. \quad (6)$$

The case $r = \infty$ is excluded and is replaced by, for any $r > 2$ there exists a constant M_r such that

$$\forall v \in W_0^{1,r}(\Omega), \quad \|v\|_{L^\infty(\Omega)} \leq \mathcal{M}_r |v|_{W^{1,r}(\Omega)}. \quad (7)$$

Owing to (5), we use the seminorm $|\cdot|_{H^1(\Omega)}$ as a norm on $H_0^1(\Omega)$ and we use it to define the norm of the dual space $H^{-1}(\Omega)$ of $H_0^1(\Omega)$:

$$\|f\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v \rangle}{|v|_{H^1(\Omega)}}.$$

In addition to the H^1 norm, it will be convenient to define the following norm with the parameter α :

$$\|v\|_\alpha = \left(\|v\|_{L^2(\Omega)}^2 + \alpha |v|_{H^1(\Omega)}^2 \right)^{1/2}.$$

In the following, we denote by $\|\cdot\|$ the L^2 norm.

We shall also use the standard space for incompressible flow:

$$H(\operatorname{div}; \Omega) = \{v \in L^2(\Omega)^2; \operatorname{div} v \in L^2(\Omega)\}$$

$$H(\operatorname{curl}; \Omega) = \{v \in L^2(\Omega)^2; \operatorname{curl} v \in L^2(\Omega)\}$$

$$V = \{v \in H_0^1(\Omega)^2; \operatorname{div} v = 0 \text{ in } \Omega\}$$

$$V^\perp = \{v \in H_0^1(\Omega)^2; \forall w \in V, (\nabla v, \nabla w) = 0\}$$

$$L_0^2(\Omega) = \{v \in L^2(\Omega); \int_\Omega q dx = 0\}$$

and the space transport:

$$X_v = \{f \in L^2(\Omega); v \cdot \nabla f \in L^2(\Omega)\},$$

where v is a given velocity in $H^1(\Omega)^2$.

2.2 Auxiliary theoretical results

To analyze, we shall use the following results. The first theorem concerns the divergence operator in any dimension d . Its proof can be found for instance in Girault and Raviart [4].

Theorem 1 *Let Ω be a bounded Lipschitz-continuous domain of \mathbb{R}^d . The divergence operator is an isomorphism from V^\perp onto $L_0^2(\Omega)$ and there exists a constant $\beta > 0$ such that for all $f \in L_0^2(\Omega)$, there exists a unique $v \in V^\perp$ satisfying*

$$\operatorname{div} v = f \quad \text{in } \Omega \quad \text{and} \quad \|v\|_{H^1(\Omega)} \leq \frac{1}{\beta} \|f\|.$$

The second result concerns the regularity of the Stokes operator in two dimensions, see [6].

Theorem 2 *Let Ω be a bounded polygon in the plane.*

1. *For each $r \in]1, 4/3[$, the Stokes operator is an isomorphism from*

$$\left[(W^{2,r}(\Omega))^2 \cap V \right] \times [W^{1,r}(\Omega) \cap L_0^2(\Omega)] \quad \text{onto} \quad L^r(\Omega)^2,$$

i.e. for each $f \in L^r(\Omega)^2$, there exists a constant C_r and a unique pair

$$(\mathbf{u}, p) \in \left[(W^{2,r}(\Omega))^2 \cap V \right] \times [W^{1,r}(\Omega) \cap L_0^2(\Omega)]$$

such that

$$-v\Delta \mathbf{u} + \nabla p = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u} = 0 \text{ on } \partial\Omega,$$

and

$$|\mathbf{u}|_{W^{2,r}(\Omega)} + |p|_{W^{1,r}(\Omega)} \leq C_r \|f\|_{L^r(\Omega)}.$$

2. *If in addition, Ω is a convex polygon, then the Stokes operator is an isomorphism from $\left[(H^2(\Omega))^2 \cap V \right] \times [H^1(\Omega) \cap L_0^2(\Omega)]$ onto $L_0^2(\Omega)^2$. Furthermore, there exists a real number $r > 2$, depending on the largest inner angle of $\partial\Omega$ such that for all $t \in [2, r]$, the Stokes operator is an isomorphism from $\left[(W^{2,t}(\Omega))^2 \cap V \right] \times [W^{1,t}(\Omega) \cap L_0^2(\Omega)]$ onto $L^t(\Omega)^2$.*

The next result concerns the unique solvability of the steady transport equation in any dimension d , see [5].

Theorem 3 *Let Ω be a bounded Lipschitz-continuous domain of \mathbb{R}^d and let \mathbf{u} be a given velocity in V .*

1. *For every f in $L^2(\Omega)$ and every constant $\gamma > 0$, the transport equation*

$$z + \gamma \mathbf{u} \cdot \nabla z = f \quad \text{in } \Omega,$$

has a unique solution $z \in X_{\mathbf{u}}$ and

$$\|z\| \leq \|f\| \tag{8}$$

2. *The following Green's formula holds:*

$$\forall z, \theta \in X_{\mathbf{u}}, \quad (\mathbf{u} \cdot \nabla z, \theta) = -(\mathbf{u} \cdot \nabla \theta, z). \tag{9}$$

Finally, the last result establishes compact embeddings in space and time. Its proof, due to Simon, see [9].

Theorem 4 (Simon) *Let X, E, Y be three Banach spaces with continuous embeddings: $X \subset E \subset Y$, the imbedding of X into E being compact. Then for any number $q \in [1, \infty]$, the space*

$$\left\{ v \in L^q(0, T; X); \frac{\partial v}{\partial t} \in L^1(0, T; Y) \right\} \tag{10}$$

is compactly imbedded in $L^q(0, T; E)$.

2.3 Formulation of the problem

Let $[0, T]$ be a time interval for some positive time T , let Ω be an domain in two dimensions, with a Lipschitz-continuous boundary $\partial\Omega$ and let \mathbf{n} denote the unit normal to $\partial\Omega$, pointing outside Ω . Let $\mathbf{f} \in L^2(0, T; H(\text{curl}; \Omega))$, the initial velocities $\mathbf{u}_0, \mathbf{h}_0 \in V$ with $\text{curl}(\mathbf{u}_0 - \alpha\Delta\mathbf{u}_0) \in L^2(\Omega)$, and we expect the velocity $\mathbf{u} \in L^\infty(0, T; V)$ with $\partial\mathbf{u}/\partial t \in L^2(0, T; V)$, the magnetic field $\mathbf{h} \in L^\infty(0, t; V)$ with $\partial\mathbf{h}/\partial t \in L^2(0, T; V)$, and the pressures $p, \omega \in L^2(0, T; L_0^2(\Omega))$.

The system (3) can be rewritten by introducing the auxiliary variable $z = \text{curl}(\mathbf{u} - \alpha\Delta\mathbf{u})$, as

$$\begin{aligned} \frac{\partial}{\partial t}(\mathbf{u} - \alpha\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + z \times \mathbf{u} - (\mathbf{h} \cdot \nabla)\mathbf{h} &= \mathbf{f} - \nabla(p^* + \mathbf{h}^2) \\ \frac{\partial\mathbf{h}}{\partial t} - \Delta\mathbf{h} + (\mathbf{u} \cdot \nabla)\mathbf{h} - (\mathbf{h} \cdot \nabla)\mathbf{u} &= -\text{grad } \omega \\ \text{div } \mathbf{u} &= \text{div } \mathbf{h} = 0 \\ \mathbf{u} = \mathbf{h} &= 0 \quad \text{on } \partial\Omega \times (0, T) \\ \mathbf{u}(0) &= \mathbf{u}_0; \quad \mathbf{h}(0) = \mathbf{h}_0 \quad \text{in } \Omega \end{aligned} \tag{11}$$

Taking the αcurl in the first of the above equations, it can be written as:

$$\alpha \frac{\partial z}{\partial t} + \nu z + \alpha(\mathbf{u} \cdot \nabla)z = \nu\text{curl}\mathbf{u} + \alpha\text{curl}(\mathbf{h} \cdot \nabla)\mathbf{h} + \alpha\text{curl}\mathbf{f} - \alpha\text{curl}\nabla(p^* + \mathbf{h}^2) \tag{12}$$

where we have used the fact that $\text{curl}(z \times \mathbf{u}) = \mathbf{u} \cdot \nabla z$, valid in two dimensions. Considering the above equation, we can rewrite the system (11) as follows:

$$\begin{aligned} \frac{\partial}{\partial t}(\mathbf{u} - \alpha\Delta\mathbf{u}) - \nu\Delta\mathbf{u} + z \times \mathbf{u} &= \mathbf{f} + (\mathbf{h} \cdot \nabla)\mathbf{h} - \nabla(p^* + \mathbf{h}^2) \\ \frac{\partial\mathbf{h}}{\partial t} - \Delta\mathbf{h} + (\mathbf{u} \cdot \nabla)\mathbf{h} - (\mathbf{h} \cdot \nabla)\mathbf{u} &= -\text{grad } \omega \\ \alpha \frac{\partial z}{\partial t} + \nu z + \alpha(\mathbf{u} \cdot \nabla)z - \nu\text{curl}\mathbf{u} &= \alpha\text{curl}(\mathbf{h} \cdot \nabla)\mathbf{h} + \alpha\text{curl}\mathbf{f} \\ &\quad - \alpha\text{curl}\nabla(p^* + \mathbf{h}^2) \end{aligned} \tag{13}$$

$$\text{div } \mathbf{u} = \text{div } \mathbf{h} = 0.$$

Semi-discretization in time

Let $N > 1$ be an integer, define the time step k by

$$k = \frac{T}{N}$$

and the subdivision points $t^n = nk$. For each $n \geq 1$, we approximate $\mathbf{f}(t^n)$ by the average defined almost everywhere in Ω by

$$\mathbf{f}^n(x) = \frac{1}{k} \int_{t^{n-1}}^{t^n} \mathbf{f}(x, s) ds.$$

We set

$$\mathbf{u}^0 = \mathbf{u}_0, \mathbf{h}^0 = \mathbf{h}_0 \text{ and } z^0 = \operatorname{curl}(\mathbf{u}_0 - \alpha \Delta \mathbf{u}_0).$$

Then, our semi-discrete problem reads: Find sequences $(\mathbf{u}^n)_{n \geq 1}, (\mathbf{h}^n)_{n \geq 1}, (z^n)_{n \geq 1}, (p^n)_{n \geq 1}$ and $(\omega^n)_{n \geq 1}$ such that $\mathbf{u}^n, \mathbf{h}^n \in V, z^n \in L^2(\Omega)$, and $p^n, \omega^n \in L_0^2(\Omega)$, solution of:

$$\begin{aligned} & \frac{1}{k} (\mathbf{u}^{n+1} - \mathbf{u}^n) - \alpha \frac{1}{k} \Delta (\mathbf{u}^{n+1} - \mathbf{u}^n) - \nu \Delta \mathbf{u}^{n+1} + z^n \times \mathbf{u}^{n+1} \\ &= \mathbf{f}^{n+1} + \mathbf{h}^{n+1} \cdot \nabla \mathbf{h}^{n+1} - \nabla (p^{*n+1} + (\mathbf{h}^{n+1})^2), \\ & \frac{1}{k} (\mathbf{h}^{n+1} - \mathbf{h}^n) - \Delta \mathbf{h}^{n+1} + \mathbf{u}^{n+1} \cdot \nabla \mathbf{h}^{n+1} - \mathbf{h}^{n+1} \cdot \nabla \mathbf{u}^{n+1} = -\operatorname{grad} \omega^{n+1}, \end{aligned} \quad (14)$$

$$\begin{aligned} & \frac{\alpha}{k} (z^{n+1} - z^n) + \nu z^{n+1} + \alpha \mathbf{u}^{n+1} \cdot \nabla z^{n+1} = \nu \operatorname{curl} \mathbf{u}^{n+1} + \alpha \operatorname{curl} \mathbf{f}^{n+1} \\ & + \alpha \operatorname{curl} (\mathbf{h}^{n+1} \cdot \nabla \mathbf{h}^{n+1}) - \alpha \operatorname{curl} \nabla (p^{*n+1} + (\mathbf{h}^{n+1})^2) \end{aligned}$$

Now, we will make some estimates for $\mathbf{u}^i, \mathbf{h}^i, z^i, p^i$ and ω^i . Multiplying the first Eq. of (14) by $2k\mathbf{u}^{n+1}$, the second Eq. of (14) by $2k\mathbf{h}^{n+1}$ and the third Eq. of (14) by $2kz^{n+1}$ and observing that $\operatorname{curl} \nabla F = 0$, for any vector field F , we obtain

$$\begin{aligned} & 2(\mathbf{u}^{i+1} - \mathbf{u}^i, \mathbf{u}^{i+1}) - 2\alpha (\Delta (\mathbf{u}^{i+1} - \mathbf{u}^i), \mathbf{u}^{i+1}) - 2k\nu (\Delta \mathbf{u}^{i+1}, \mathbf{u}^{i+1}) \\ &= 2k(\mathbf{f}^{i+1}, \mathbf{u}^{i+1}) + 2k(\mathbf{h}^{i+1} \cdot \nabla \mathbf{h}^{i+1}, \mathbf{u}^{i+1}), \\ & 2(\mathbf{h}^{i+1} - \mathbf{h}^i, \mathbf{h}^{i+1}) - 2k(\Delta \mathbf{h}^{i+1}, \mathbf{h}^{i+1}) = 2k(\mathbf{h}^{i+1} \cdot \nabla \mathbf{u}^{i+1}, \mathbf{h}^{i+1}), \\ & 2(z^{i+1} - z^i, z^{i+1}) + \frac{2\nu k}{\alpha} (z^{i+1}, z^{i+1}) = 2k(\operatorname{curl} \mathbf{f}^{i+1}, z^{i+1}) \\ & + \frac{2\nu k}{\alpha} (\operatorname{curl} \mathbf{u}^{i+1}, z^{i+1}) + 2k(\operatorname{curl} (\mathbf{h}^{i+1} \cdot \nabla \mathbf{h}^{i+1}), z^{i+1}), \end{aligned} \quad (15)$$

where we used that fact that $(\mathbf{u}^{i+1} \cdot \mathbf{h}^{i+1}, \mathbf{h}^{i+1}) = 0$.

Proposition 5 *The sequence $(\mathbf{u}^n)_{n \geq 1}$ and $(\mathbf{h}^n)_{n \geq 1}$ satisfy the following uniform a priori estimates:*

$$\begin{aligned} & \sum_{i=0}^{n-1} k \|\nabla \mathbf{u}^{i+1}\|_{\alpha}^2 \leq \frac{C^2 \overline{C}}{\nu^2} \|\mathbf{f}\|_{L^2(\Omega, \times]0, t^n])}^2 + \frac{1}{\nu} \|\mathbf{u}_0\|_{\alpha}^2 + \frac{1}{\nu} \|\mathbf{h}_0\|^2, \\ & \sum_{i=0}^{n-1} k \|\nabla \mathbf{h}^{i+1}\|^2 \leq \frac{C^2 \overline{C}}{2\nu} \|\mathbf{f}\|_{L^2(\Omega \times]0, t^n])}^2 + \frac{1}{2} \|\mathbf{u}_0\|_{\alpha}^2 + \frac{1}{2} \|\mathbf{h}_0\|^2, \\ & 2\|\nabla \mathbf{h}^{i+1}\|^2 + \sum_{j=1}^i \left(\|\nabla \mathbf{h}^{j+1} - \nabla \mathbf{h}^j\|^2 + \frac{k}{\mu\sigma} \|A\mathbf{h}^{j+1}\|^2 \right) \leq \|\nabla \mathbf{h}_0\|^2. \end{aligned} \quad (16)$$

Proof: Multiplying the first Eq. of (14) by $2\mathbf{u}^{i+1}$ and the second Eq. of (14) by $2\mathbf{h}^{i+1}$, we obtain

$$\begin{aligned} & \frac{2}{k} (\mathbf{u}^{i+1} - \mathbf{u}^i, \mathbf{u}^{i+1}) - \alpha \frac{2}{k} \Delta (\mathbf{u}^{i+1} - \mathbf{u}^i, \mathbf{u}^{i+1}) - 2\nu (\Delta \mathbf{u}^{i+1}, \mathbf{u}^{i+1}) \\ &= 2 (\mathbf{f}^{i+1}, \mathbf{u}^{i+1}) + 2 (\mathbf{h}^{i+1} \cdot \nabla \mathbf{h}^{i+1}, \mathbf{u}^{i+1}), \\ & \frac{2}{k} (\mathbf{h}^{i+1} - \mathbf{h}^i, \mathbf{h}^{i+1}) - 2 (\Delta \mathbf{h}^{i+1}, \mathbf{h}^{i+1}) - 2 (\mathbf{h}^{i+1} \cdot \nabla \mathbf{u}^{i+1}, \mathbf{h}^{i+1}) = 0 \end{aligned}$$

where we should note that

$$\begin{aligned} (z^i \times \mathbf{u}^{i+1}, \mathbf{u}^{i+1}) &= 0, & \left(\nabla \left(p^{*i+1} + (\mathbf{h}^{i+1})^2 \right), \mathbf{u}^{i+1} \right) &= 0, \\ (\text{grad } \omega^{i+1}, \mathbf{h}^{i+1}) &= 0, & (\mathbf{u}^{i+1} \cdot \nabla \mathbf{h}^{i+1}, \mathbf{h}^{i+1}) &= 0. \end{aligned}$$

Using the formula

$$2(a - b, a) = \|a\|^2 - \|b\|^2 + \|a - b\|^2, \quad (17)$$

that is true in any Hilbert space, and adding the above equations, adding from $i = 0$ to $n - 1$ and making use the telescopic property, we have

$$\begin{aligned} & \frac{1}{k} \|\mathbf{u}^n\|_\alpha^2 + \frac{1}{k} \|\mathbf{h}^n\|^2 + \frac{1}{k} \sum_{i=0}^{n-1} [\|\mathbf{u}^{i+1} - \mathbf{u}^i\|_\alpha^2] + \frac{1}{k} \sum_{i=0}^{n-1} \|\mathbf{h}^{i+1} - \mathbf{h}^i\|^2 \\ &+ 2\nu \sum_{i=0}^{n-1} \|\nabla \mathbf{u}^{i+1}\|^2 + 2 \sum_{i=0}^{n-1} \|\nabla \mathbf{h}^{i+1}\|^2 \\ &\leq 2C \sum_{i=0}^{n-1} \|\mathbf{f}^{i+1}\| \|\mathbf{u}^{i+1}\| + \frac{1}{k} \|\mathbf{u}_0\|_\alpha^2 + \frac{1}{k} \|\mathbf{h}_0\|^2. \end{aligned} \quad (18)$$

Now taking into account that

$$\begin{aligned} & 2C \|\mathbf{f}^{i+1}\| \|\mathbf{u}^{i+1}\| \leq \\ & 4C^2 \frac{\delta}{2} \|\mathbf{f}^{i+1}\|^2 + \frac{1}{2\delta} \|\mathbf{u}^{i+1}\|^2 \leq 4C^2 \frac{\delta}{2} \|\mathbf{f}^{i+1}\|^2 + \frac{\overline{C}}{2\delta} \|\nabla \mathbf{u}^{i+1}\|^2, \end{aligned}$$

then from equation (18) we can write

$$\begin{aligned} & 2\nu \sum_{i=0}^{n-1} \|\nabla \mathbf{u}^{i+1}\|^2 + 2 \sum_{i=0}^{n-1} \|\nabla \mathbf{h}^{i+1}\|^2 \leq \\ & \sum_{i=0}^{n-1} \left(4C^2 \frac{\delta}{2} \|\mathbf{f}^{i+1}\|^2 + \frac{\overline{C}}{2\delta} \|\nabla \mathbf{u}^{i+1}\|^2 \right) + \frac{1}{k} \|\mathbf{u}_0\|_\alpha^2 + \frac{1}{k} \|\mathbf{h}_0\|^2 \end{aligned}$$

then, putting $\delta = \overline{C}/2\nu$, we obtain

$$\nu \sum_{i=0}^{n-1} \|\nabla \mathbf{u}^{i+1}\|^2 + 2 \sum_{i=0}^{n-1} \|\nabla \mathbf{h}^{i+1}\|^2 \leq \frac{C^2 \overline{C}}{\nu k} \|\mathbf{f}\|_{L^2(\Omega, \times]0, t^n])}^2 + \frac{1}{k} \|\mathbf{u}_0\|_\alpha^2 + \frac{1}{k} \|\mathbf{h}_0\|^2.$$

On the other hand, to obtain estimates of the $\|\nabla \mathbf{h}^{i+1}\|^2$, we multiply the second equation in (14) by $2A\mathbf{h}^{i+1}$, then we obtain (after applying the projection operator P)

$$\frac{2}{k} (\nabla \mathbf{h}^{i+1} - \nabla \mathbf{h}^i, \nabla \mathbf{h}^{i+1}) + \frac{2}{\mu\sigma} \|A\mathbf{h}^{i+1}\|^2 = -2(\mathbf{u}^{i+1} \cdot \nabla \mathbf{h}^{i+1}, A\mathbf{h}^{i+1}) + 2(\mathbf{h}^{i+1} \cdot \nabla \mathbf{u}^{i+1}, A\mathbf{h}^{i+1}),$$

then bounded each of terms, we have

$$\frac{2}{k} (\nabla \mathbf{h}^{i+1} - \nabla \mathbf{h}^i, \nabla \mathbf{h}^{i+1}) = \frac{1}{k} \|\nabla \mathbf{h}^{i+1}\|^2 - \frac{1}{k} \|\nabla \mathbf{h}^i\|^2 + \frac{1}{k} \|\nabla \mathbf{h}^{i+1} - \nabla \mathbf{h}^i\|^2,$$

$$\begin{aligned} |2(\mathbf{u}^{i+1} \cdot \nabla \mathbf{h}^{i+1}, A\mathbf{h}^{i+1})| &\leq 2\|\mathbf{u}^{i+1}\|_{L^6}^2 \|\nabla \mathbf{h}^{i+1}\|_{L^3}^2 \|A\mathbf{h}^{i+1}\|^2 \\ &\leq 2\|\nabla \mathbf{u}^{i+1}\|^2 \|\nabla \mathbf{h}^{i+1}\|^{1/2} \|A\mathbf{h}^{i+1}\|^{3/2} \\ &\leq 2C_{\varepsilon_1} \|\nabla \mathbf{u}^{i+1}\|^4 \|\nabla \mathbf{h}^{i+1}\|^2 + 2\varepsilon_1 \|A\mathbf{h}^{i+1}\|^2, \end{aligned}$$

$$\begin{aligned} |2(\mathbf{h}^{i+1} \cdot \nabla \mathbf{u}^{i+1}, A\mathbf{h}^{i+1})| &\leq 2\|\mathbf{h}^{i+1}\|_{L^\infty} \|\nabla \mathbf{u}^{i+1}\| \|A\mathbf{h}^{i+1}\| \\ &\leq 2C \|\nabla \mathbf{h}^{i+1}\|^{1/2} \|A\mathbf{h}^{i+1}\|^{3/2} \|\nabla \mathbf{u}^{i+1}\| \\ &\leq 2CC_{\varepsilon_2} \|\nabla \mathbf{u}^{i+1}\|^4 \|\nabla \mathbf{h}^{i+1}\|^2 + 2C\varepsilon_2 \|A\mathbf{h}^{i+1}\|^2, \end{aligned}$$

where we use the estimate of interpolation $\|\mathbf{h}^{i+1}\|_{L^\infty} \leq C \|\nabla \mathbf{h}^{i+1}\|^{1/2} \|A\mathbf{h}^{i+1}\|^{1/2}$. From above estimates and taking into account that $\|\nabla \mathbf{u}^{i+1}\|$ is bounded, we have

$$\begin{aligned} &\|\nabla \mathbf{h}^{i+1}\|^2 + \|\nabla \mathbf{h}^{i+1} - \nabla \mathbf{h}^i\|^2 + \frac{2k}{\mu\sigma} \|A\mathbf{h}^{i+1}\|^2 \\ &\leq k(2C_{\varepsilon_1} + 2CC_{\varepsilon_2}) \overline{C} \|\nabla \mathbf{h}^{i+1}\|^2 + k(2\varepsilon_1 + 2C\varepsilon_2) \|A\mathbf{h}^{i+1}\|^2 + \|\nabla \mathbf{h}^i\|^2 \end{aligned}$$

then, there is k and ε such that (for a sufficiently large N) $1 - k(2C_{\varepsilon_1} + 2CC_{\varepsilon_2}) \overline{C} = 1/2$ and $2k/\mu\sigma - k(2\varepsilon_1 + 2C\varepsilon_2) = k/\mu\sigma$, thus, from the above inequality we can write

$$2\|\nabla \mathbf{h}^{i+1}\|^2 + \|\nabla \mathbf{h}^{i+1} - \nabla \mathbf{h}^i\|^2 + \frac{k}{\mu\sigma} \|A\mathbf{h}^{i+1}\|^2 \leq \|\nabla \mathbf{h}^i\|^2,$$

from which we get (using the lemma 3.14 pg. 131 in [11] with $\eta = \gamma_i = \xi = 0$)

$$2\|\nabla \mathbf{h}^{i+1}\|^2 + \sum_{j=1}^i \left(\|\nabla \mathbf{h}^{j+1} - \nabla \mathbf{h}^j\|^2 + \frac{k}{\mu\sigma} \|A\mathbf{h}^{j+1}\|^2 \right) \leq \|\nabla \mathbf{h}_0\|^2.$$

□

Proposition 6 *The sequence $(\mathbf{u}^n)_{n \geq 1}$ and $(\mathbf{h}^n)_{n \geq 1}$ satisfy the following uniform a priori estimates, for $1 \leq n \leq N$*

$$\|\mathbf{u}^n\|_\alpha^2 + \sum_{i=0}^{n-1} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|_\alpha^2 \leq \frac{C^2}{2\nu} \|\mathbf{f}\|_{L^2(\Omega \times (0, t^n))}^2 + \|\mathbf{u}_0\|_\alpha^2 + \|\mathbf{h}_0\|^2, \quad (19)$$

$$\|\mathbf{h}^n\|^2 + \sum_{i=0}^{n-1} \|\mathbf{h}^{i+1} - \mathbf{h}^i\|^2 \leq \frac{C^2}{2\nu} \|\mathbf{f}\|_{L^2(\Omega \times (0, t^n))}^2 + \|\mathbf{u}_0\|_\alpha^2 + \|\mathbf{h}_0\|^2, \quad (20)$$

Proof: Estimates (19) and (20) are derived by adding the first and second equations of (15) and using the formula (17),

$$\begin{aligned}
& \|\mathbf{u}^{i+1}\|^2 - \|\mathbf{u}^i\|^2 + \|\mathbf{u}^{i+1} - \mathbf{u}^i\|^2 + \|\mathbf{h}^{i+1}\|^2 \\
& - \|\mathbf{h}^i\|^2 + \|\mathbf{h}^{i+1} - \mathbf{h}^i\|^2 - 2\alpha \left(\Delta (\mathbf{u}^{i+1} - \mathbf{u}^i), \mathbf{u}^{i+1} - \mathbf{u}^i \right) \\
& - 2\alpha \left(\Delta (\mathbf{u}^{i+1} - \mathbf{u}^i), \mathbf{u}^i \right) - 2k\nu \|\nabla \mathbf{u}^{i+1}\|^2 + 2k \|\nabla \mathbf{h}^{i+1}\|^2 = 2k (\mathbf{f}^{i+1}, \mathbf{u}^{i+1})
\end{aligned}$$

then adding from $i = 0$ to $i = n - 1$ and making use of the telescopic property, and again using the formulae (17), we have

$$\begin{aligned}
& \|\mathbf{u}^n\|_\alpha^2 - \|\mathbf{u}_0\|_\alpha^2 + \sum_{i=0}^{n-1} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|_\alpha^2 + \|\mathbf{h}^n\|^2 \\
& - \|\mathbf{h}_0\|^2 + \sum_{i=0}^{n-1} \|\mathbf{h}^{i+1} - \mathbf{h}^i\|^2 + 2\alpha \sum_{i=0}^{n-1} \|\nabla (\mathbf{u}^{i+1} - \mathbf{u}^i)\|^2 \\
& + 2k \sum_{i=0}^{n-1} \|\nabla \mathbf{h}^{i+1}\|^2 \leq \frac{C^2}{2\nu} \sum_{i=0}^{n-1} k \|\mathbf{f}^{i+1}\|^2,
\end{aligned} \tag{21}$$

now, drooping $2\alpha \sum_{i=0}^{n-1} \|\nabla (\mathbf{u}^{i+1} - \mathbf{u}^i)\|^2$ and $2k \sum_{i=0}^{n-1} \|\nabla \mathbf{h}^{i+1}\|^2$ we have

$$\begin{aligned}
& \|\mathbf{u}^n\|_\alpha^2 + \sum_{i=0}^{n-1} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|_\alpha^2 + \|\mathbf{h}^n\|^2 + \sum_{i=0}^{n-1} \|\mathbf{h}^{i+1} - \mathbf{h}^i\|^2 \\
& \leq \frac{C^2}{2\nu} \|\mathbf{f}\|_{L^2(\Omega, \times]0, t^n])}^2 + \|\mathbf{u}_0\|_\alpha^2 + \|\mathbf{h}_0\|^2,
\end{aligned}$$

where $\sum_{i=0}^{n-1} k \|\mathbf{f}^{i+1}\|^2 = \|\mathbf{f}\|_{L^2(\Omega, \times]0, t^n])}^2$. From which we get the result. □

Remark 7 From (21) we have

$$2\alpha \sum_{i=0}^{n-1} \|\nabla (\mathbf{u}^{i+1} - \mathbf{u}^i)\|^2 + 2k \sum_{i=0}^{n-1} \|\nabla \mathbf{h}^n\|^2 \leq \frac{C^2}{2\nu} \|\mathbf{f}\|_{L^2(\Omega, \times]0, t^n])}^2 + \|\mathbf{u}_0\|_\alpha^2 + \|\mathbf{h}_0\|^2. \tag{22}$$

Estimate for z^n

Before obtaining an estimate for z^n , we will show the following corollary

Corollary 8 Give the sequence $(\mathbf{h}^n)_{n \geq 1}$ we have the following uniform a priori estimates

$$\|\mathbf{A}\mathbf{h}^n\| \leq C \quad \text{and} \quad \|\nabla (\mathbf{h}^n \cdot \nabla \mathbf{h}^n)\| \leq C \quad \forall n \in \mathbb{N}.$$

Proof: Applying the projector P to the second equation of (14) we obtain

$$A\mathbf{h}^{n+1} = -P(\mathbf{u}^{n+1} \cdot \nabla \mathbf{h}^{n+1}) + P(\mathbf{h}^{n+1} \cdot \nabla \mathbf{u}^{n+1}) - \frac{1}{k}P(\mathbf{h}^{n+1} - \mathbf{h}^n)$$

then

$$\|A\mathbf{h}^{n+1}\| \leq \|\mathbf{u}^{n+1} \cdot \nabla \mathbf{h}^{n+1}\| + \|\mathbf{h}^{n+1} \cdot \nabla \mathbf{u}^{n+1}\| + \frac{1}{k}\|\mathbf{h}^{n+1} - \mathbf{h}^n\| \quad (23)$$

thus, each term can be estimated as follows:

a)

$$\begin{aligned} \|\mathbf{u}^{n+1} \cdot \nabla \mathbf{h}^{n+1}\| &\leq \|\mathbf{u}^{n+1}\|_{L^6(\Omega)} \|\nabla \mathbf{h}^{n+1}\|_{L^3(\Omega)} \\ &\leq C \|\nabla \mathbf{u}^{n+1}\| \|\nabla \mathbf{h}^{n+1}\|^{1/2} \|A\mathbf{h}^{n+1}\|^{1/2} \\ &\leq C \|A\mathbf{h}^{n+1}\|^{1/2} \\ &\leq C_{\varepsilon_1} + \varepsilon_1 \|A\mathbf{h}^{n+1}\| \text{ for } \varepsilon_1 > 0 \text{ small} \end{aligned}$$

here was used $H^1 \hookrightarrow L^6$ and a result of interpolation.

b)

$$\begin{aligned} \|\mathbf{h}^{n+1} \cdot \nabla \mathbf{u}^{n+1}\| &\leq \|\mathbf{h}^{n+1}\|_{L^\infty(\Omega)} \|\nabla \mathbf{u}^{n+1}\| \\ &\leq C \|\mathbf{h}^{n+1}\|_{L^\infty(\Omega)} \\ &\leq C \|\mathbf{h}^{n+1}\|^{1/2} \|A\mathbf{h}^{n+1}\|^{1/2} \\ &\leq C \|A\mathbf{h}^{n+1}\|^{1/2} \\ &\leq C_{\varepsilon_2} + \varepsilon_2 \|A\mathbf{h}^{n+1}\|. \end{aligned}$$

Finally, substituting in (23) we obtain

$$(1 - \varepsilon_1 - \varepsilon_2) \|A\mathbf{h}^{n+1}\| \leq C_{\varepsilon_1} + C_{\varepsilon_2} + C$$

and considering $(1 - \varepsilon_1 - \varepsilon_2) > 0$ we obtain

$$\|A\mathbf{h}^{n+1}\| \leq C, \quad (24)$$

where the constant C is generic.

Now, taking into account the equation (24) and usual estimates, we have

$$\begin{aligned} &\|\nabla(\mathbf{h}^{n+1} \cdot \nabla \mathbf{h}^{n+1})\| \\ &\leq C_1 \|\nabla \mathbf{h}^{n+1} \cdot \nabla \mathbf{h}^{n+1}\| + C_2 \|\mathbf{h}^{n+1} \cdot \nabla^2 \mathbf{h}^{n+1}\| \\ &\leq C_3 \|\nabla \mathbf{h}^{n+1}\|_{L^3(\Omega)} \|\nabla \mathbf{h}^{n+1}\|_{L^6(\Omega)} + C_4 \|\mathbf{h}^{n+1}\|_{L^\infty(\Omega)} \|\nabla^2 \mathbf{h}^{n+1}\| \\ &\leq C_5 \|A\mathbf{h}^{n+1}\| \|A\mathbf{h}^{n+1}\| + C_6^{1/2} \|A\mathbf{h}^{n+1}\|^{1/2} \|A\mathbf{h}^{n+1}\| \\ &\leq C, \end{aligned}$$

another consequence of (24) is $\|\mathbf{h}^n \cdot \nabla \mathbf{h}^{n+1}\| \leq C$. □

Proposition 9 *The sequence $(z^n)_{n \geq 1}$ satisfy the following uniform a priori estimates*

$$\begin{aligned} & \|z^n\|^2 + \sum_{i=0}^{n-1} \|z^{i+1} - z^i\|^2 \leq \\ & \leq \left(\frac{C^2 \overline{C}}{\alpha} \|\mathbf{f}\|_{L^2(\Omega, \times]0, t^n])}^2 + \frac{\nu}{2\alpha} \|\mathbf{u}_0\|_\alpha^2 + \frac{1}{\alpha} \|\mathbf{h}_0\| \right) \\ & + \frac{2\alpha k}{\nu} \|\operatorname{curl} \mathbf{f}\|_{L^2(\Omega \times (0, t^n))}^2 + \frac{2\alpha}{\nu} CT + \|z_0\|^2. \end{aligned} \quad (25)$$

Proof: Multiplying the third equation of (14) by $\frac{2k}{\alpha} z^{i+1}$, we get

$$\begin{aligned} & 2(z^{i+1} - z^i, z^{i+1}) + \frac{2k\nu}{\alpha} (z^{i+1}, z^{i+1}) = \frac{2k\nu}{\alpha} (\operatorname{curl} \mathbf{u}^{i+1}, z^{i+1}) \\ & + 2k (\operatorname{curl} \mathbf{f}^{i+1}, z^{i+1}) + 2k (\operatorname{curl} (\mathbf{h}^{i+1} \cdot \nabla \mathbf{h}^{i+1}), z^{i+1}) \end{aligned}$$

then, using the formula (17), we obtain

$$\begin{aligned} & \|z^{i+1}\|^2 - \|z^i\|^2 + \|z^{i+1} - z^i\|^2 + \frac{2k\nu}{\alpha} \|z^{i+1}\|^2 \\ & \leq \frac{2k\nu}{\alpha} \|\nabla \mathbf{u}^{i+1}\| \|z^{i+1}\| + 2k \|\operatorname{curl} \mathbf{f}^{i+1}\| \|z^{i+1}\| \\ & + 2k \|\nabla (\mathbf{h}^{i+1} \cdot \nabla \mathbf{h}^{i+1})\| \|z^{i+1}\| \\ & \leq \frac{2k\nu}{\alpha} \left(C_{\varepsilon_1} \|\nabla \mathbf{u}^{i+1}\|^2 + \varepsilon_1^2 \|z^{i+1}\|^2 \right) + 2k C_{\varepsilon_2} \|\operatorname{curl} \mathbf{f}^{i+1}\|^2 \\ & + 2k \varepsilon_2 \|z^{i+1}\|^2 + 2k C \|z^{i+1}\| \\ & \leq \frac{2k\nu}{2\alpha \varepsilon_1} \|\nabla \mathbf{u}^{i+1}\|^2 + \frac{k\nu \varepsilon_1}{\alpha} \|z^{i+1}\|^2 + 2 \frac{2k}{2\varepsilon_2} \|\operatorname{curl} \mathbf{f}^{i+1}\|^2 \\ & + k \varepsilon_2 \|z^{i+1}\|^2 + \frac{k}{\varepsilon_3} + k \varepsilon_3 \|z^{i+1}\|^2. \end{aligned}$$

Now adding to $i = 0$ to $n - 1$ we obtain

$$\begin{aligned} & \sum_{i=0}^{n-1} \left(\|z^{i+1}\|^2 - \|z^i\|^2 \right) + \sum_{i=0}^{n-1} \|z^{i+1} - z^i\|^2 \\ & + \left(\frac{2k\nu}{\alpha} - \frac{k\nu}{\alpha} \varepsilon_1 - k \varepsilon_2 - k C \varepsilon_3 \right) \sum_{i=0}^{n-1} \|z^{i+1}\|^2 \\ & \leq \frac{k\nu}{\alpha \varepsilon_1} \sum_{i=0}^{n-1} \|\nabla \mathbf{u}^{i+1}\|^2 + \frac{k}{\varepsilon_2} \sum_{i=0}^{n-1} \|\operatorname{curl} \mathbf{f}^{i+1}\|^2 + C \frac{T}{\varepsilon_3}. \end{aligned}$$

Where the term $\frac{k}{\varepsilon_3} \sum_{i=0}^{n-1} C = \frac{k}{\varepsilon_3} Cn \leq \frac{k}{\varepsilon_3} C \frac{T}{k} = C \frac{T}{\varepsilon_3}$. Now considering $\varepsilon_1 = 1$ and $\varepsilon_2 = \varepsilon_3 = \nu/2\alpha$ the above equation can be written as

$$\begin{aligned} & \sum_{i=0}^{n-1} \left(\|z^{i+1}\|^2 - \|z^i\|^2 \right) + \sum_{i=0}^{n-1} \|z^{i+1} - z^i\|^2 \\ & \leq \frac{k\nu}{\alpha} \sum_{i=0}^{n-1} \|\nabla \mathbf{u}^{i+1}\|^2 + \frac{2\alpha k}{\nu} \sum_{i=0}^{n-1} \|\operatorname{curl} \mathbf{f}^{i+1}\|^2 + \frac{2\alpha}{\nu} CT, \end{aligned}$$

consequently,

$$\begin{aligned} \|z^n\|^2 + \sum_{i=0}^{n-1} \|z^{i+1} - z^i\|^2 & \leq \left(\frac{C^2 \overline{C}}{\alpha} \|\mathbf{f}\|_{L^2(\Omega, \times]0, t^n])}^2 + \frac{\nu}{2\alpha} \|\mathbf{u}_0\|_\alpha^2 + \frac{1}{\alpha} \|\mathbf{h}_0\| \right) \\ & + \frac{2\alpha k}{\nu} \|\operatorname{curl} \mathbf{f}\|_{L^2(\Omega \times (0, t^n))}^2 + \frac{2\alpha}{\nu} CT + \|z_0\|^2. \end{aligned}$$

□

Proposition 10 *Let*

$$C_Z = \sup_{0 \leq n \leq N-1} \|z^n\|^2.$$

The sequences $((\mathbf{u}^{n+1} - \mathbf{u}^n)/k)_{n \geq 1}$ and $((\mathbf{h}^{n+1} - \mathbf{h}^n)/k)_{n \geq 1}$, satisfy the following uniform a priori estimates:

$$\begin{aligned} \sum_{i=0}^{n-1} \frac{1}{2k} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|_\alpha^2 & \leq \frac{(C_1\nu + C_z S_4^2)^2}{2\alpha} \left(\frac{C^2 \overline{C}}{\nu^2} \|\mathbf{f}\|_{L^2(\Omega, \times]0, t^n])}^2 \right. \\ & \left. + \frac{1}{\nu} \|\mathbf{u}_0\|_\alpha^2 + \frac{1}{\nu} \|\mathbf{h}_0\|^2 \right) + C_2^2 \|\mathbf{f}\|_{L^2(\Omega \times]0, t^n])}^2 + DT, \\ \sum_{i=0}^{n-1} \frac{1}{2k} \|\mathbf{h}^{i+1} - \mathbf{h}^i\|^2 & \leq \frac{(C_1\nu + C_z S_4^2)^2}{2\alpha} \left(\frac{C^2 \overline{C}}{\nu^2} \|\mathbf{f}\|_{L^2(\Omega, \times]0, t^n])}^2 \right) \\ & + \frac{1}{\nu} \|\mathbf{u}_0\|_\alpha^2 + \frac{1}{\nu} \|\mathbf{h}_0\|^2 \Big) + C_2^2 \|\mathbf{f}\|_{L^2(\Omega, \times]0, t^n])}^2 + DT. \end{aligned}$$

Proof: Multiplying the first Eq. (14) by $(\mathbf{u}^{i+1} - \mathbf{u}^i)$ and the second Eq. of (14) by $(\mathbf{h}^{i+1} - \mathbf{h}^i)$, we obtain

$$\begin{aligned} & \frac{1}{k} (\mathbf{u}^{i+1} - \mathbf{u}^i, \mathbf{u}^{i+1} - \mathbf{u}^i) - \frac{\alpha}{k} \Delta (\mathbf{u}^{i+1} - \mathbf{u}^i, \mathbf{u}^{i+1} - \mathbf{u}^i) - \nu (\Delta \mathbf{u}^{i+1}, \mathbf{u}^{i+1} - \mathbf{u}^i) \\ & + (z^i \times \mathbf{u}^{i+1}, \mathbf{u}^{i+1} - \mathbf{u}^i) = (\mathbf{f}^{i+1}, \mathbf{u}^{i+1} - \mathbf{u}^i) + (\mathbf{h}^{i+1} \cdot \nabla \mathbf{h}^{i+1}, \mathbf{u}^{i+1} - \mathbf{u}^i), \\ & \frac{1}{k} (\mathbf{h}^{i+1} - \mathbf{h}^i, \mathbf{h}^{i+1} - \mathbf{h}^i) - (\Delta \mathbf{h}^{i+1}, \mathbf{h}^{i+1} - \mathbf{h}^i) + (\mathbf{u}^{i+1} \cdot \nabla \mathbf{h}^{i+1}, \mathbf{h}^{i+1} - \mathbf{h}^i) \\ & - (\mathbf{h}^{i+1} \cdot \nabla \mathbf{u}^{i+1}, \mathbf{h}^{i+1} - \mathbf{h}^i) = 0, \end{aligned} \tag{26}$$

then adding the above equations and using that

$$|(z^i \times \mathbf{u}^{i+1}, \mathbf{u}^{i+1} - \mathbf{u}^i)| \leq C_z S_4^2 \|\mathbf{u}^{i+1}\|_{H^1(\Omega)} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|_{H^1(\Omega)},$$

we have

$$\begin{aligned} & \frac{1}{k} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|_\alpha^2 + \frac{1}{k} \|\mathbf{h}^{i+1} - \mathbf{h}^i\|^2 \\ & \leq (C_1 \nu + C_z S_4^2)^2 \frac{\varepsilon_1}{2\alpha} \|\mathbf{u}^{i+1}\|_{H^1(\Omega)}^2 + \frac{\alpha}{2\varepsilon_1} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|_{H^1(\Omega)}^2 \\ & \quad + \frac{C_2^2 \varepsilon_2}{2} \|\mathbf{f}^{i+1}\|^2 + \frac{1}{2\varepsilon_2} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|^2 + \frac{C_3^2 k \varepsilon_3}{2} \|\mathbf{h}^{i+1} \cdot \nabla \mathbf{h}^{i+1}\|^2 \\ & \quad + \frac{1}{2k\varepsilon_3} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|^2 + C_4^2 k \frac{\varepsilon_4}{2} \|A\mathbf{h}^{i+1}\|^2 \\ & \quad + \frac{1}{2k\varepsilon_4} \|\mathbf{h}^{i+1} - \mathbf{h}^i\|^2 + C_5 k \frac{\varepsilon_5}{2} \|\mathbf{u}^{i+1} \cdot \nabla \mathbf{h}^{i+1}\|^2 \\ & \quad + \frac{1}{2k\varepsilon_5} \|\mathbf{h}^{i+1} - \mathbf{h}^i\|^2 + \frac{C_6^2 k \varepsilon_6}{2} \|\mathbf{h}^{i+1} \cdot \nabla \mathbf{u}^{i+1}\|^2 + \frac{1}{2k\varepsilon_6} \|\mathbf{h}^{i+1} - \mathbf{h}^i\|^2, \end{aligned}$$

then put $\varepsilon_1 = k, \varepsilon_2 = 2k, \varepsilon_3 = 2, \varepsilon_4 = 2, \varepsilon_5 = 4$ and $\varepsilon_7 = 4$, of the above equation can be written

$$\begin{aligned} & \frac{1}{2k} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|_\alpha^2 + \frac{1}{2k} \|\mathbf{h}^{i+1} - \mathbf{h}^i\| \\ & \leq (C_1 \nu + C_z S_4^2)^2 \frac{k}{2\alpha} \|\mathbf{u}^{i+1}\|_{H^1(\Omega)}^2 + C_2^2 k \|\mathbf{f}^{i+1}\| + C_3 k \|\mathbf{h}^{i+1} \cdot \nabla \mathbf{h}^{i+1}\|^2 \\ & \quad + C_4^2 k \|A\mathbf{h}^{i+1}\|^2 + 2C_5^2 k \|\mathbf{u}^{i+1} \cdot \nabla \mathbf{h}^{i+1}\|^2 + 2C_6^2 k \|\mathbf{h}^{i+1} \cdot \nabla \mathbf{u}^{i+1}\|^2. \end{aligned} \tag{27}$$

On the other hand, recalling the Corollary 7, we have

$$\begin{aligned} \|\mathbf{h}^{i+1} \cdot \nabla \mathbf{h}^{i+1}\|^2 & \leq d_1, \quad \|A\mathbf{h}^{i+1}\|^2 \leq d_2 \\ \|\mathbf{u}^{i+1} \cdot \nabla \mathbf{h}^{i+1}\|^2 & \leq d_3, \quad \|\mathbf{h}^{i+1} \cdot \nabla \mathbf{u}^{i+1}\|^2 \leq d_4 \end{aligned}$$

and summing from $i = 0$ to $n - 1$ in (27), we obtain

$$\begin{aligned} & \sum_{i=0}^{n-1} \frac{1}{2k} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|_\alpha^2 + \sum_{i=0}^{n-1} \frac{1}{2k} \|\mathbf{h}^{i+1} - \mathbf{h}^i\| \\ & \leq \sum_{i=0}^{n-1} (C_1 \nu + C_z S_4^2)^2 \frac{k}{2\alpha} \|\mathbf{u}^{i+1}\|_{H^1(\Omega)}^2 + C_2^2 \sum_{i=0}^{n-1} k \|\mathbf{f}^{i+1}\|^2 \\ & \quad + \sum_{i=0}^{n-1} Dk, \end{aligned}$$

where $D = C_3^2 d_1 + C_4^2 d_2 + 2C_5^2 d_3 + 2C_6^2 d_4$, then observed $n \leq N = T/k$ we can write $\sum_{i=0}^{n-1} Dk = Dnk \leq DT$, then from the above inequality we obtain

$$\begin{aligned} & \sum_{i=0}^{n-1} \frac{1}{2k} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|_\alpha^2 + \sum_{i=0}^{n-1} \frac{1}{2k} \|\mathbf{h}^{i+1} - \mathbf{h}^i\| \\ & \leq \sum_{i=0}^{n-1} (C_1 \nu + C_z S_4^2)^2 \frac{k}{2\alpha} \|\mathbf{u}^{i+1}\|_{H^1(\Omega)}^2 + C_2^2 \sum_{i=0}^{n-1} k \|\mathbf{f}^{i+1}\|^2 + DT. \end{aligned}$$

Indeed, from (16) we obtain

$$\begin{aligned} & \sum_{i=0}^{n-1} \frac{1}{2k} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|_\alpha^2 + \sum_{i=0}^{n-1} \frac{1}{2k} \|\mathbf{h}^{i+1} - \mathbf{h}^i\| \\ & \leq \frac{(C_1 \nu + C_z S_4^2)^2}{\alpha} \left(\frac{C^2 \overline{C}}{\nu^2} \|\mathbf{f}\|_{L^2(\Omega, \times]0, t^n])}^2 + \frac{1}{\nu} \|\mathbf{u}_0\|_\alpha^2 + \frac{1}{\nu} \|\mathbf{h}_0\|^2 \right) \\ & \quad + 2C_2^2 \|\mathbf{f}\|_{L^2(\Omega, \times]0, t^n])}^2 + 2DT. \end{aligned} \tag{28}$$

From which we get the result. \square

Proposition 11 *The sequence $(p^n)_{n \geq 1}$ and $(\omega^n)_{n \geq 1}$ satisfy the following uniform a priori estimates:*

$$\begin{aligned} & \sum_{i=0}^{n-1} \left[k \|p^{i+1}\|^2 + k \|\omega^{i+1}\|^2 \right] \leq \\ & \left[\frac{(C_1 \nu + C_z S_4^2)^2 L}{\alpha} + 4C_z^2 S_4^4 \right] \left(\frac{C^2 \overline{C}}{\nu^2} \|\mathbf{f}\|_{L^2(\Omega, \times]0, t^n])}^2 + \frac{1}{\nu} \|\mathbf{u}_0\|_\alpha^2 + \frac{1}{\nu} \|\mathbf{h}_0\|^2 \right) \\ & + [4C_6^2 C + 2C_2^2 L] \|\mathbf{f}\|^2 + [\overline{D} + 2LD]T, \quad 1 \leq n \leq N. \end{aligned}$$

Proof: Let $v_1, v_2 \in V^\perp = \{v \in H_0^1(\Omega); \forall w \in V, (\nabla v, \nabla w) = 0\}$ such that $\operatorname{div} v_1 = p^{i+1}$ and $\operatorname{grad} v_2 = \omega^{i+1}$. Then by multiplying the first and second eq.(14) by v_1 and v_2 respectively, and adding the results, we have

$$\begin{aligned} & \frac{1}{k} (\mathbf{u}^{i+1} - \mathbf{u}^i, v_1) + (z^i \times \mathbf{u}^{i+1}, v_1) + (p^{i+1}, p^{i+1}) \\ & - (\mathbf{h}^{i+1} \cdot \nabla \mathbf{h}^{i+1}, v_1) + \frac{1}{k} (\mathbf{h}^{i+1} - \mathbf{h}^i, v_2) + (\mathbf{u}^{i+1} \cdot \nabla \mathbf{h}^{i+1}, v_2) \\ & - (\mathbf{h}^{i+1} \cdot \nabla \mathbf{u}^{i+1}, v_2) + (\omega^{i+1}, \omega^{i+1}) = (\mathbf{f}^{i+1}, v_1). \end{aligned}$$

where, we consider that $v_1, v_2 \in V^\perp$. Thus, we can write

$$\begin{aligned}
& k \|p^{i+1}\|^2 + k \|\omega^{i+1}\|^2 \leq C_1 \frac{\delta_1}{2} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|^2 + \frac{1}{2\delta_1} \|v_1\|^2 \\
& + k C_z^2 S_4^4 \frac{\delta_2}{2} \|\mathbf{u}^{i+1}\|_{H^1}^2 + \frac{k}{2\delta_2} \|v_1\|_{H^1}^2 + k C_2^2 \frac{\delta_3}{2} \|\mathbf{h}^{i+1} \cdot \nabla \mathbf{h}^{i+1}\|^2 \\
& + \frac{k}{2\delta_3} \|v_1\|^2 + C_3^2 \frac{\delta_4}{2} \|\mathbf{h}^{i+1} - \mathbf{h}^i\|^2 + \frac{1}{2\delta_4} \|v_2\|^2 \\
& + k C_4^2 \frac{\delta_5}{2} \|\mathbf{u}^{i+1} \cdot \nabla \mathbf{h}^{i+1}\|^2 + \frac{k}{2\delta_5} \|v_2\|^2 + k C_5^2 \frac{\delta_6}{2} \|\mathbf{h}^{i+1} \cdot \nabla \mathbf{u}^{i+1}\|^2 \\
& + \frac{k}{2\delta_6} \|v_2\|^2 + k C_6^2 \frac{\delta_7}{2} \|\mathbf{f}^{i+1}\|^2 + \frac{k}{2\delta_7} \|v_1\|^2.
\end{aligned}$$

Now, we considering that $H^1 \hookrightarrow L^2$, i.e.,

$$\|v_1\| \leq C \|\nabla v_1\| = C \|p^{i+1}\|$$

and

$$\|v_2\| \leq C \|\nabla v_2\| = C \|w^{i+1}\|.$$

From which, we obtain

$$\begin{aligned}
& k \|p^{i+1}\|^2 + k \|\omega^{i+1}\|^2 \leq C_1 \frac{\delta_1}{2} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|^2 + \frac{C}{2\delta_1} \|p^{i+1}\|^2 \\
& + k C_z^2 S_4^4 \frac{\delta_2}{2} \|\mathbf{u}^{i+1}\|_{H^1}^2 + \frac{k}{2\delta_2} \|p^{i+1}\|^2 + k C_2^2 \frac{\delta_3}{2} \|\mathbf{h}^{i+1} \cdot \nabla \mathbf{h}^{i+1}\|^2 \\
& + \frac{Ck}{2\delta_3} \|p^{i+1}\|^2 + C_3^2 \frac{\delta_4}{2} \|\mathbf{h}^{i+1} - \mathbf{h}^i\|^2 + \frac{C}{2\delta_4} \|\omega^{i+1}\|^2 \\
& + k C_4^2 \frac{\delta_5}{2} \|\mathbf{u}^{i+1} \cdot \nabla \mathbf{h}^{i+1}\|^2 + \frac{Ck}{2\delta_5} \|\omega^{i+1}\|^2 + k C_5^2 \frac{\delta_6}{2} \|\mathbf{h}^{i+1} \cdot \nabla \mathbf{u}^{i+1}\|^2 \\
& + \frac{Ck}{2\delta_6} \|\omega^{i+1}\|^2 + k C_6^2 \frac{\delta_7}{2} \|\mathbf{f}^{i+1}\|^2 + \frac{Ck}{2\delta_7} \|p^{i+1}\|^2,
\end{aligned}$$

then, taking $\delta_1 = 4C/k$, $\delta_2 = 4$, $\delta_3 = \delta_7 = 4C$ and $\delta_4 = 2C/k$, $\delta_5 = \delta_6 = 4C$, addyng from $i = 0$ to $n - 1$ and multiplying by 2, we have

$$\begin{aligned}
& \sum_{i=0}^{n-1} \left[k \|p^{i+1}\|^2 + k \|w^{i+1}\|^2 \right] \leq 4C_1^2 C \sum_{i=0}^{n-1} \frac{1}{k} \|\mathbf{u}^{i+1} - \mathbf{u}^i\|^2 \\
& + 4C_z^2 S_4^4 \sum_{i=0}^{n-1} k \|\mathbf{u}^{i+1}\|_{H^1}^2 + 2C_3^2 C \sum_{i=0}^{n-1} \frac{1}{k} \|\mathbf{h}^{i+1} - \mathbf{h}^i\|^2 + 4C_6^2 C \|\mathbf{f}\|_{L^2(\Omega, [0, T])}^2 \\
& + \sum_{i=0}^{n-1} (4C_2^2 C d_1 + 4C_4^2 C d_3 + 4C_5^2 C d_4) k,
\end{aligned}$$

thus, put $\overline{D} = 4C_2^2Cd_1 + 4C_4^2Cd_3 + 4C_5^2Cd_4$, noting that $n \leq T/k$ and using the inequalities (28) and (16), we have

$$\begin{aligned} & \sum_{i=0}^{n-1} \left[k \|p^{i+1}\|^2 + k \|\omega^{i+1}\|^2 \right] \leq \\ & \leq \left[\frac{(C_1\nu + C_z S_4^2)^2 L}{\alpha} + 4C_z^2 S_4^4 \right] \left(\frac{C^2 \overline{C}}{\nu^2} |\mathbf{f}|_{L^2(\Omega \times]0, t^n])}^2 + \frac{1}{\nu} \|\mathbf{u}_0\|_\alpha^2 + \frac{1}{\nu} \|\mathbf{h}_0\|^2 \right) \\ & + [4C_6^2 C + 2C_2^2 L] |\mathbf{f}|_{L^2(\Omega \times]0, T])}^2 + [\overline{D} + 2LD] T. \end{aligned}$$

where $L = \max \{4C_1^2 C, 4C_1^2 C\}$ □

2.4 Existence of solutions

Here, it is convenient to transform the sequence $(\mathbf{u}^n), (\mathbf{h}^n), (p^n), (\omega^n)$ and (z^n) into functions. Since $(\mathbf{u}^n), (\mathbf{h}^n)$ and (z^n) need to be differentiated, we define the piecewise linear functions in time:

$$\forall t \in [t^n, t^{n+1}], \quad \mathbf{u}_k(t) = \mathbf{u}^n + \frac{t - t^n}{k} (\mathbf{u}^{n+1} - \mathbf{u}^n), \quad 0 \leq n \leq N-1$$

$$\forall t \in [t^n, t^{n+1}], \quad \mathbf{h}_k(t) = \mathbf{h}^n + \frac{t - t^n}{k} (\mathbf{h}^{n+1} - \mathbf{h}^n), \quad 0 \leq n \leq N-1$$

$$\forall t \in [t^n, t^{n+1}], \quad z_k(t) = z^n + \frac{t - t^n}{k} (z^{n+1} - z^n), \quad 0 \leq n \leq N-1.$$

Next, in view of the other terms in (14), we define the step functions:

$$\forall t \in [t^n, t^{n+1}] \quad \text{and} \quad 0 \leq n \leq N-1;$$

$$\mathbf{f}_k(t) = \mathbf{f}^{n+1}, \quad \mathbf{w}_k(t) = \mathbf{u}^{n+1}, \quad \mathbf{g}_k(t) = \mathbf{h}^{n+1}, \quad p_k(t) = p^{n+1},$$

$$\omega_k(t) = \omega^{n+1}, \quad \zeta_k(t) = z^{n+1}, \quad \lambda_k(t) = z^n.$$

Then we have the following convergences.

Proposition 12 *The exist functions $\mathbf{u}, \mathbf{h} \in L^\infty(0, T; V)$ with $\partial \mathbf{u} / \partial t, \partial \mathbf{h} / \partial t \in L^2(0, T; V), p, \omega \in L^2(0, T; L_0^2(\Omega))$ and $z \in L^\infty(0, T; L^2(\Omega))$ such that a subsequence of k , still denoted by k , satis-*

fies:

$$\lim_{k \rightarrow 0} \mathbf{u}_k = \lim_{k \rightarrow 0} \mathbf{w}_k = \mathbf{u} \quad \text{weakly } * \text{ in } L^\infty(0, T; V),$$

$$\lim_{k \rightarrow 0} \mathbf{h}_k = \lim_{k \rightarrow 0} \mathbf{g}_k = \mathbf{h} \quad \text{weakly } * \text{ in } L^\infty(0, T; V),$$

$$\lim_{k \rightarrow 0} z_k = \lim_{k \rightarrow 0} \zeta_k = \lim_{k \rightarrow 0} \lambda_k = z \quad \text{weakly } * \text{ in } L^\infty(0, T; L^2(\Omega)),$$

$$\lim_{k \rightarrow 0} p_k = p \quad \text{weakly in } L^2(0, T; L_0^2(\Omega)),$$

$$\lim_{k \rightarrow 0} \omega_k = \omega \quad \text{weakly in } L^2(0, T; L_0^2(\Omega)),$$

$$\lim_{k \rightarrow 0} \frac{\partial}{\partial t} \mathbf{u}_k = \frac{\partial}{\partial t} \mathbf{u} \quad \text{weakly in } L^2(0, T; V),$$

$$\lim_{k \rightarrow 0} \frac{\partial}{\partial t} \mathbf{h}_k = \frac{\partial}{\partial t} \mathbf{h} \quad \text{weakly in } L^2(0, T; V).$$

Furthermore,

$$\begin{aligned} \lim_{k \rightarrow 0} \mathbf{u}_k &= \lim_{k \rightarrow 0} \mathbf{w}_k = \mathbf{u} \quad \text{strongly in } L^\infty(0, T; L^4(\Omega)^2), \\ \lim_{k \rightarrow 0} \mathbf{h}_k &= \lim_{k \rightarrow 0} \mathbf{g}_k = \mathbf{h} \quad \text{strongly in } L^\infty(0, T; L^4(\Omega)^2) \end{aligned} \tag{29}$$

Proof: Due to the uniform estimates given in Propositions 5 -10, we can extract a subsequence (still denoted by k) such that:

$$\lim_{k \rightarrow 0} \mathbf{u}_k = \mathbf{u}; \quad \lim_{k \rightarrow 0} \mathbf{h}_k = \mathbf{h} \quad \text{weakly } * \text{ in } L^\infty(0, T; V),$$

$$\lim_{k \rightarrow 0} z_k = z \quad \text{weakly } * \text{ in } L^\infty(0, T; L^2(\Omega)),$$

$$\lim_{k \rightarrow 0} p_k = p; \quad \lim_{k \rightarrow 0} w_k = w \quad \text{weakly in } L^2(0, T; L_0^2(\Omega)),$$

$$\lim_{k \rightarrow 0} \frac{\partial}{\partial t} \mathbf{u}_k = \frac{\partial}{\partial t} \mathbf{u}; \quad \lim_{k \rightarrow 0} \frac{\partial}{\partial t} \mathbf{h}_k = \frac{\partial}{\partial t} \mathbf{h} \quad \text{weakly in } L^2(0, T; V),$$

$$\lim_{k \rightarrow 0} \mathbf{w}_k = \mathbf{w}; \quad \lim_{k \rightarrow 0} g_k = g \quad \text{weakly } * \text{ in } L^\infty(0, T; V),$$

$$\lim_{k \rightarrow 0} \zeta_k = \zeta; \quad \lim_{k \rightarrow 0} \lambda_k = \lambda \quad \text{weakly in } L^2(0, T; L^2(\Omega)).$$

As far as the function $\mathbf{w}, \mathbf{g}, \zeta$ and λ are concerned, observe that

$$\forall t \in [t^n, t^{n+1}], \quad \mathbf{w}_k(t) - \mathbf{u}_k(t) = \frac{t^{n+1} - t}{k} (\mathbf{u}^{n+1} - \mathbf{u}^n), \quad 0 \leq n \leq N-1$$

$$\forall t \in [t^n, t^{n+1}], \quad \mathbf{g}_k(t) - \mathbf{h}_k(t) = \frac{t^{n+1} - t}{k} (\mathbf{h}^{n+1} - \mathbf{h}^n), \quad 0 \leq n \leq N-1$$

$$\forall t \in [t^n, t^{n+1}], \quad \zeta_k(t) - z_k(t) = \frac{t^{n+1} - t}{k} (z^{n+1} - z^n), \quad 0 \leq n \leq N-1$$

$$\forall t \in [t^n, t^{n+1}], \quad \lambda_k(t) - z_k(t) = \frac{t^{n+1} - t}{k} (z^{n+1} - z^n), \quad 0 \leq n \leq N-1.$$

Therefore

$$\begin{aligned} \|\mathbf{w}_k - \mathbf{u}_k\|_{L^2(0,T;V)}^2 &= \frac{k}{3} \sum_{n=0}^{N-1} \|\mathbf{u}^{n+1} - \mathbf{u}^n\|_{H^1(\Omega)}^2, \\ \|\mathbf{g}_k - \mathbf{h}_k\|^2 &= \frac{k}{3} \sum_{n=0}^{N-1} \|\mathbf{h}^{n+1} - \mathbf{h}^n\|_{H^1(\Omega)}^2, \end{aligned} \quad (30)$$

$$\|\zeta_k - z_k\|_{L^2(\Omega \times]0,T])}^2 = \|\lambda_k - z_k\|_{L^2(\Omega \times]0,T])}^2 = \frac{k}{3} \sum_{n=0}^{N-1} \|z^{n+1} - z^n\|^2.$$

Then, using the estimates (19),(20),(25) and the uniqueness of the limit, we have $\mathbf{w} = \mathbf{u}, \mathbf{g} = \mathbf{h}$ and $\zeta = \lambda = z$. It remains to prove the strong convergence (29). In view of (30), it suffices to prove the strong convergence of \mathbf{u}_k and \mathbf{h}_k . Note that, (\mathbf{u}_k) and (\mathbf{h}_k) are bounded uniformly in the space

$$\left\{ \mathbf{v} \in L^2(0,T; H_0^1(\Omega)^2); \frac{\partial \mathbf{v}}{\partial t} \in L^2(0,T; L^4(\Omega)^2) \right\},$$

and as the imbedding of $H^1(\Omega)$ into $L^4(\Omega)$ is compact, the Simon's theorem implies that \mathbf{u}_k and \mathbf{h}_k converges strongly to \mathbf{u} and \mathbf{h} respectively in $L^2(0,T; L^4(\Omega)^2)$. \square

Theorem 13 *Let Ω be a bounded Lipschitz-continuous domain in two dimensions. Then for any $\alpha > 0, v > 0, \mathbf{f} \in L^2(0,T; H(\text{curl}; \Omega))$ and $\mathbf{u}_0, \mathbf{h}_0 \in V$ with $\text{curl}(\mathbf{u}_0 - \alpha \Delta \mathbf{u}_0) \in L^2(\Omega)$, problem (11) has at least one solution $\mathbf{u}, \mathbf{h} \in L^\infty(0,T; V)$ with $\frac{\partial \mathbf{u}}{\partial t}, \frac{\partial \mathbf{h}}{\partial t} \in L^2(0,T; V)$ and $p, \omega \in L^2(0,T; L_0^2(\Omega))$.*

Proof: Let k be a subsequence satisfying the convergences of above Proposition. It is easy to check that the functions $\mathbf{u}_k, \mathbf{h}_k, z_k, p_k, \omega_k, \mathbf{w}_k, \mathbf{g}_k, \zeta_k$ and λ_k satisfy the following formulations:

$$\begin{aligned} \forall \mathbf{v} \in H_0^1(\Omega), \forall \varphi \in \mathcal{C}^0([0, T]), \quad & \int_0^T \left[\left(\frac{\partial}{\partial t} \mathbf{u}_k(t), \mathbf{v} \right) + \alpha \left(\frac{\partial}{\partial t} \nabla \mathbf{u}_k(t), \nabla \mathbf{v} \right) \right. \\ & + v(\nabla \mathbf{w}_k(t), \nabla \mathbf{v}) + (\lambda_k(t) \times \mathbf{w}_k(t), \mathbf{v}) - (p_k(t), \operatorname{div} \mathbf{v})] \varphi(t) dt \\ & + \int_0^T (\mathbf{g}_k(t) \cdot \nabla \mathbf{v}, g_k(t)) \varphi(t) dt = \int_0^T (\mathbf{f}_k(t), \mathbf{v}) \varphi(t) dt, \end{aligned} \quad (31)$$

$$\begin{aligned} \forall \mathbf{g} \in H_0^1(\Omega), \forall \phi \in \mathcal{C}^0([0, T]), \quad & \int_0^T \left[\left(\frac{\partial}{\partial t} \mathbf{h}_k(t), \mathbf{g} \right) + (\nabla \mathbf{g}_k(t), \nabla \mathbf{g}) \right. \\ & - (\mathbf{w}_k(t) \cdot \nabla \mathbf{g}, \mathbf{g}_k(t)) + (\mathbf{g}_k(t) \cdot \nabla \mathbf{g}, \mathbf{w}_k(t)) - (\mathbf{w}_k, \operatorname{div} \mathbf{g})] \phi(t) dt = 0, \end{aligned} \quad (32)$$

$$\begin{aligned} \forall \theta \in W^{1,4}(\Omega), \forall \psi \in \mathcal{C}^1([0, T]) \text{ with } \psi(T) = 0, \\ -\alpha \int_0^T (z_k(t), \theta) \frac{\partial}{\partial t} \psi(t) dt + \int_0^T [v(\zeta_k(t), \theta) - \alpha(\mathbf{w}_k(t) \cdot \nabla \theta, \zeta_k(t))] \psi(t) dt \\ - (z^0, \theta) \psi(0) = \int_0^T [v(\operatorname{curl} \mathbf{w}_k(t), \theta) + \alpha(\operatorname{curl} \mathbf{f}_k(t), \theta)] \psi(t) dt \\ + \int_0^T \alpha(\operatorname{curl}(\mathbf{g}_k(t) \cdot \nabla \mathbf{g}_k(t)), \theta) \psi(t) dt, \end{aligned} \quad (33)$$

where we note that

$$\frac{\partial}{\partial t} \mathbf{u}_k = \frac{\partial}{\partial t} \left[\mathbf{u}^n + \frac{t - t^n}{k} (\mathbf{u}^{n+1} - \mathbf{u}^n) \right] = \frac{1}{k} (\mathbf{u}^{n+1} - \mathbf{u}^n)$$

Note that, the weak convergences of the proposition above, imply the convergences of all the linear terms in (31), (32) and (33) and the terms involving \mathbf{f} also converge, from standard integration results. Thus, it suffices to check the convergence of the non-linear terms. Then, for all indices i and j , $1 \leq i, j \leq 2$,

$$\lim_{k \rightarrow 0} (w_k)_i v_j \varphi = u_i v_j \varphi; \quad \lim_{k \rightarrow 0} (g_k)_i g_j \phi = h_i g_j \phi \quad \text{strongly in } L^2(\Omega \times]0, T[)$$

and

$$\lim_{k \rightarrow 0} \lambda_k = z \quad \text{weakly in } L^2(\Omega \times]0, T[),$$

then, we have

$$\begin{aligned}
\lim_{k \rightarrow 0} \int_0^T (\lambda_k(t) \times \mathbf{w}_k(t), \mathbf{v}) \varphi(t) dt &= \int_0^T (z(t) \times \mathbf{u}(t), \mathbf{v}) \varphi(t) dt, \\
\lim_{k \rightarrow 0} \int_0^T (\mathbf{g}_k(t) \cdot \nabla \mathbf{v}, \mathbf{g}_k(t)) \varphi(t) dt &= \int_0^T (\mathbf{h}(t) \cdot \nabla \mathbf{v}, \mathbf{h}(t)) \varphi(t) dt, \\
\lim_{k \rightarrow 0} \int_0^T (\mathbf{w}_k(t) \cdot \nabla \mathbf{g}, \mathbf{g}_k(t)) \phi(t) dt &= \int_0^T (\mathbf{u}(t) \cdot \nabla \mathbf{g}, \mathbf{h}(t)) \phi(t) dt, \\
\lim_{k \rightarrow 0} \int_0^T (\mathbf{g}_k(t) \cdot \nabla \mathbf{g}, \mathbf{w}_k(t)) \phi(t) dt &= \int_0^T (\mathbf{h}(t) \cdot \nabla \mathbf{g}, \mathbf{u}(t)) \phi(t) dt.
\end{aligned}$$

Similary

$$\begin{aligned}
\lim_{k \rightarrow 0} (\mathbf{w}_k \cdot \nabla \theta) \psi &= (\mathbf{u} \cdot \nabla \theta) \psi \quad \text{strongly in } L^2(\Omega \times]0, T[), \\
\lim_{k \rightarrow 0} (\mathbf{g}_k \cdot \nabla \text{curl} \theta) \psi &= (\mathbf{h} \cdot \nabla \text{curl} \theta) \psi \quad \text{strongly in } L^2(\Omega \times]0, T[)
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{k \rightarrow 0} \int_0^T (\mathbf{w}_k \cdot \nabla \theta, \zeta_k) \psi(t) dt &= \int_0^T (\mathbf{u} \cdot \nabla \theta, z(t)) \psi(t) dt, \\
\lim_{k \rightarrow 0} \int_0^T (\mathbf{g}_k \cdot \nabla \text{curl} \theta, \mathbf{g}_k) \psi(t) dt &= \int_0^T (\mathbf{h} \cdot \nabla \text{curl} \theta, \mathbf{h}) \psi(t) dt.
\end{aligned}$$

Hence we can pass to the limit in (31), (32) and (33) and we obtain

$$\begin{aligned}
\forall \mathbf{v} \in H_0^1(\Omega), \forall \varphi \in \mathcal{C}^0([0, T]), \quad &\int_0^T \left[\left(\frac{\partial}{\partial t} \mathbf{u}(t), \mathbf{v} \right) + \alpha \left(\frac{\partial}{\partial t} \nabla \mathbf{u}(t), \nabla \mathbf{v} \right) \right. \\
&+ v(\nabla \mathbf{u}(t), \nabla \mathbf{v}) + (z(t) \times \mathbf{u}(t), \mathbf{v}) - (p(t), \text{div} \mathbf{v})] \varphi(t) dt \\
&+ \int_0^T (\mathbf{h}(t) \cdot \nabla \mathbf{v}, \mathbf{h}(t)) \varphi(t) dt = \int_0^T (\mathbf{f}(t), \mathbf{v}) \varphi(t) dt, \\
\forall \mathbf{g} \in H_0^1(\Omega), \forall \phi \in \mathcal{C}^0([0, T]), \quad &\int_0^T \left[\left(\frac{\partial}{\partial t} \mathbf{h}(t), \mathbf{g} \right) + (\nabla \mathbf{h}(t), \nabla \mathbf{g}) \right. \\
&- (\mathbf{u}(t) \cdot \nabla \mathbf{g}, \mathbf{h}(t)) + (\mathbf{h}(t) \cdot \nabla \mathbf{g}, \mathbf{u}(t)) - (\omega, \text{div} \mathbf{g})] \phi(t) dt = 0, \\
\forall \theta \in W^{1,4}(\Omega), \forall \psi \in \mathcal{C}^1([0, T]) \quad &\text{with } \psi(T) = 0, \\
&-\alpha \int_0^T (z(t), \theta) \frac{\partial}{\partial t} \psi(t) dt + \int_0^T [v(z(t), \theta) - \alpha(\mathbf{u}(t) \cdot \nabla \theta, z(t))] \psi(t) dt \\
&- (z^0, \theta) \psi(0) = \int_0^T [v(\text{curl} \mathbf{u}(t), \theta) + \alpha(\text{curl} \mathbf{f}(t), \theta)] \psi(t) dt \\
&+ \int_0^T \alpha(\text{curl}(\mathbf{h}(t) \cdot \nabla \mathbf{h}(t)), \theta) \psi(t) dt.
\end{aligned}$$

By choosing $\mathbf{v}, \mathbf{g} \in \mathcal{D}(\Omega)^2$, φ, ϕ and $\psi \in \mathcal{D}(]0, T[)$ and $\theta \in \mathcal{D}(\Omega)$, we easily recover (13). It remain to recover the initial data, for this note for any $\mathbf{g} \in L^2(\Omega)^2$ and any $\phi \in H^1(0, T)$ satisfying $\phi(T) = 0$, and used the formula $\mathbf{h}_k(t) = \mathbf{h}^n + \frac{t - t^n}{k} (\mathbf{h}^{n+1} - \mathbf{h}^n)$ we have

$$\int_0^T \left(\frac{\partial}{\partial t} \mathbf{h}_k(t), \mathbf{g} \right) \phi(t) dt = - \int_0^T (\mathbf{h}_k(t), \mathbf{g}) \frac{\partial}{\partial t} \phi(t) dt - (\mathbf{h}^0, \mathbf{g}) \varphi(0)$$

Passing to the limit in the equality above, we have

$$\int_0^T \left(\frac{\partial}{\partial t} \mathbf{h}(t), \mathbf{g} \right) \phi(t) dt = - \int_0^T (\mathbf{h}(t), \mathbf{g}) \frac{\partial}{\partial t} \phi(t) dt - (\mathbf{h}^0, \mathbf{g}) \varphi(0).$$

On the other hand, we have

$$\int_0^T \left(\frac{\partial}{\partial t} \mathbf{h}(t), \mathbf{g} \right) \phi(t) dt = - \int_0^T (\mathbf{h}(t), \mathbf{g}) \frac{\partial}{\partial t} \phi(t) dt - (\mathbf{h}(0), \mathbf{g}) \varphi(0)$$

where we conclude that $\mathbf{h}^0 = \mathbf{h}(0)$, similarly we obtain $\mathbf{u}^0 = \mathbf{u}(0)$ and $z^0 = z(0)$. \square

With respect to the uniqueness it is possible to show an analogous to the [3]. In fact, we have

Theorem 14 *Assume that Ω is a convex polygon. Then for any $\alpha > 0, v > 0$, \mathbf{f} in $L^2(0, T; H(\text{curl}; \Omega))$ and $\mathbf{u}_0 \in V, \mathbf{h} \in V$ with $\text{curl}(\mathbf{u}_0 - \alpha \Delta \mathbf{u}_0) \in L^2(\Omega)$, problem (3)-(4) has exactly one solution $(\mathbf{u}, \mathbf{h}, p, \omega) \in W$.*

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